Reduced-Order $\mathscr{H}_2/\mathscr{H}_\infty$ Control of Discrete-Time LPV Systems with Experimental Validation on an Overhead Crane Test Setup*

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Abstract— **This paper presents a numerically attractive approach to design reduced-order multi-objective $\mathscr{H}_2/\mathscr{H}_\infty$ controllers for discrete-time linear parameter-varying (LPV) systems. The proposed controller synthesis approach relies on an a priori computed polynomially parameter-dependent full-order LPV controller that stabilizes the LPV system for all possible parameter trajectories. This full-order controller is subsequently used in a sufficient linear matrix inequality (LMI) optimization problem for reduced-order $\mathscr{H}_2/\mathscr{H}_\infty$ LPV synthesis. Pólya relaxations are used to obtain tractable LMI formulations, and a simplicial subdivision of the parameter domain is applied to relieve the numerical burden. Experimental validations on a lab-scale overhead crane with varying cable length illustrate the practical viability of the approach.**

I. INTRODUCTION

Bridging the gap between the restricted class of linear time-invariant systems and the general class of nonlinear systems, the modern framework of linear parameter-varying systems has gained popularity since the nineties, and has been successful in many application fields [1].

Full-order dynamic output feedback control, amongst others, received substantial attention in the field of LPV control [2], [3], [4], [5], [6]. Reduced-order LPV control design, on the other hand, has not been extensively studied and applied yet. For instance, the approaches presented in [7], [8] solely provide conditions for controller orders larger or equal to the plant order minus the number of exactly measured states, while the algorithms for fixed-order synthesis presented in [9], [10], which are very successful for LTI systems, cannot handle LPV dynamics. At the same time, the approach [11] relies on the design of numerous random initial controllers, which are subsequently used for the design of a single LPV controller. This results in a numerically costly procedure for the computation of a suitable LPV controller. Moreover, the latter approach does not allow all system matrices to be parameter-dependent. Lastly, although the 2-step approach presented in [12] (using a stabilizing state feedback for a specific augmented system as a starting point) is readily extendable to handle LPV dynamics, no guidelines are provided on how to select an initial state feedback to obtain a performant reduced-order controller.

In this paper, the recently developed approach [13], [14] for the design of reduced-order LTI controllers is extended to handle discrete-time LPV dynamics. The resulting approach handles any prefixed controller order, allows polynomial parameter dependencies of all system matrices, can consider multiple design objectives, and provides intuitive guidelines for the selection of an initial controller. Pólya relaxations are used to obtain tractable LMI formulations, and a simplicial subdivision of the parameter domain is applied to relieve the numerical burden without increasing conservatism. The combination of all properties above makes our approach attractive, both computationally and practically, compared to existing approaches. To illustrate the latter, our approach is experimentally validated on an overhead crane test setup with varying cable length.

The paper is organized as follows. First, Section II discusses the mathematical problem formulation. Then, the reduced-order synthesis approach is presented in Section III, followed by numerical and experimental validations in Section IV. Finally, the conclusions are given in Section V.

Notation: I_n denotes the identity matrix of dimension n and $0_{m\times n}$ denotes a zero matrix of dimension $m\times n$. The subscripts are omitted when the dimensions can be inferred from the context. The transpose of a matrix X is written as X', and the notation $\mathrm{He}\{X\} = X + X'$ is used. The sets of real symmetric (real positive definite) matrices of dimension n are denoted by $\mathbb{S}^n$ ($\mathbb{S}^n_+$). A star ($\star$) indicates symmetric terms in matrix inequalities. The expectation operator is defined as $\mathbb{E}(\cdot)$. To improve readability, the time dependency of the scheduling parameter $\alpha \in \mathbb{R}^N$, $N \in \mathbb{N}_+$, on the time index k is omitted whenever possible, by introducing the shorthand notation $\alpha := \alpha(k)$, $\alpha_+ := \alpha(k+1)$.

II. PROBLEM FORMULATION

We consider the discrete-time finite-dimensional LPV state-space realization

$$\left\{\begin{array}{rcl} x(k+1) &=& A(\alpha)x(k)+B_w(\alpha)w(k)+B_u(\alpha)u(k),\\ z(k) &=& C_z(\alpha)x(k)+D_w(\alpha)w(k)+D_u(\alpha)u(k),\\ y(k) &=& C_y(\alpha)x(k)+D_y(\alpha)w(k), \end{array}\right. \tag{1}$$

$k \geq 0$, with state $x \in \mathbb{R}^{n_x}$, exogenous input $w \in \mathbb{R}^{n_w}$, control input $u \in \mathbb{R}^{n_u}$, regulated output $z \in \mathbb{R}^{n_z}$ and measured output $y \in \mathbb{R}^{n_y}$. All system matrices are assumed to be bounded

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for $k \geq 0$ and have a polynomial parameter dependency on the scheduling parameter α that takes values in $\mathbb{R}^N$. The parameter rate of variation is defined as $\Delta\alpha := \alpha_+ - \alpha$. Accordingly, the set of admissible parameter trajectories is defined as

$$\mathscr{T} := \left\{ \alpha(\cdot) : \mathbb{N} \mapsto \mathbb{R}^N \mid (\alpha(k), \Delta\alpha(k)) \in \Lambda,\, k \geq 0 \right\}, \quad (2)$$

where $\Lambda \subset \mathbb{R}^{2N}$ is a bounded polytopic set.

The objective is to design a reduced-order LPV controller

$$C^{(q)} : \begin{cases} x_c(k+1) &= A_c(\alpha)x_c(k) + B_c(\alpha)y(k), \\ u(k) &= C_c(\alpha)x_c(k) + D_c(\alpha)y(k), \end{cases} \quad (3)$$

with $x_c \in \mathbb{R}^q$, $q < n_x$, that stabilizes the LPV system (1) and satisfy multiple closed-loop performance specifications for all parameter trajectories $\alpha(\cdot) \in \mathscr{T}$. Grouping the controller matrices of (3) as

$$\Theta(\alpha) := \begin{bmatrix} A_c(\alpha) & B_c(\alpha) \\ C_c(\alpha) & D_c(\alpha) \end{bmatrix} \in \mathbb{R}^{(q+n_u)\times(q+n_y)} \quad (4)$$

the closed-loop interconnection of the LPV system (1) with the LPV controller (3) is indicated as

$$H_\Theta : \begin{cases} x_{cl}(k+1) &= \mathscr{A}_\Theta(\alpha)x_{cl}(k) + \mathscr{B}_\Theta(\alpha)w(k), \\ z(k) &= \mathscr{C}_\Theta(\alpha)x_{cl}(k) + \mathscr{D}_\Theta(\alpha)w(k). \end{cases} \quad (5)$$

where $x_{cl} = \begin{bmatrix} x' & x_c' \end{bmatrix}' \in \mathbb{R}^{n_x+q}$ is a closed-loop state vector. Defining the matrices

$$\begin{bmatrix} \tilde{A}(\alpha) & \tilde{B}_w(\alpha) & \tilde{B}_u(\alpha) \\ \tilde{C}_z(\alpha) & \tilde{D}_w(\alpha) & \tilde{D}_u(\alpha) \\ \tilde{C}_y(\alpha) & \tilde{D}_y(\alpha) & 0 \end{bmatrix} := \left[\begin{array}{cc|c|cc} A(\alpha) & 0 & B_w(\alpha) & 0 & B_u(\alpha) \\ 0 & 0 & 0 & I_q & 0 \\ \hline C_z(\alpha) & 0 & D_w(\alpha) & 0 & D_u(\alpha) \\ \hline 0 & I_q & 0 & 0 & 0 \\ C_y(\alpha) & 0 & D_y(\alpha) & 0 & 0 \end{array}\right], \quad (6)$$

the affine dependency of the closed-loop matrices of (5) on the controller parameter is expressed as

$$\begin{bmatrix} \mathscr{A}(\alpha) & \mathscr{B}(\alpha) \\ \mathscr{C}(\alpha) & \mathscr{D}(\alpha) \end{bmatrix} = \begin{bmatrix} \tilde{A}(\alpha) & \tilde{B}_w(\alpha) \\ \tilde{C}_z(\alpha) & \tilde{D}_w(\alpha) \end{bmatrix} + \begin{bmatrix} \tilde{B}_u(\alpha) \\ \tilde{D}_u(\alpha) \end{bmatrix} \Theta(\alpha) \begin{bmatrix} \tilde{C}_y(\alpha) & \tilde{D}_y(\alpha) \end{bmatrix}. \quad (7)$$

III. REDUCED-ORDER LPV CONTROL

This section presents a novel approach to design reduced-order $\mathscr{H}_2/\mathscr{H}_\infty$ LPV controllers. This approach relies on a full-order $\mathscr{H}_2/\mathscr{H}_\infty$ LPV controller, which can be computed using, for instance, the convex approaches discussed in [4], [2]. For reasons of compactness, only the results related to $\mathscr{H}_\infty$ performance are discussed. The corresponding $\mathscr{H}_2$ characterizations follow as a straightforward extension, see also [14], [13].

First an extended analysis LMI is presented, forming an insightful starting point for the derivation of reduced-order synthesis conditions. An upper bound on the $\mathscr{H}_\infty$ performance of the closed-loop LPV system (5) can be calculated using the LMI presented in Theorem 1, which is an extension of the recently developed extended $\mathscr{H}_\infty$ performance characterization presented in [13].

Theorem 1: Let $\Psi(\alpha) \in \mathbb{R}^{(n_x+n_u)\times(n_x+n_y)}$ be an arbitrary parameter-dependent matrix, and let $\Theta_\mathrm{a}(\alpha) \in \mathbb{R}^{(n_x+n_u)\times(n_x+n_y)}$ be constructed from $\Theta(\alpha) \in \mathbb{R}^{(q+n_u)\times(q+n_y)}$ by adding Schur stable uncontrollable and/or unobservable dynamics. Then, the closed-loop system H_Θ, defined as in (5), is exponentially stable and $\|H_\Theta\|_\infty^2 < \gamma$ if there exist bounded matrices $P(\alpha) \in \mathbb{S}_+^{2n_x}$, $X_1(\alpha) \in \mathbb{R}^{2n_x\times(n_x+n_u)}$, $X_2(\alpha) \in \mathbb{R}^{n_w\times(n_x+n_u)}$ and $X_3(\alpha) \in \mathbb{R}^{(n_x+n_u)\times(n_x+n_u)}$ such that the LMI (8) (see next page) holds for all parameter trajectories $\alpha(\cdot) \in \mathscr{T}$.

Proof: The proof is based on application of the projection lemma [15], and is an extension of the proof that is presented in [13] for LTI systems. ■

Remark 1: Eliminating X_j, $j = 1, \ldots, 3$ in the LMI (8) results in the standard $\mathscr{H}_\infty$ performance characterization for discrete-time LPV systems [16], [17].

Theorem 2 (Reduced-order $\mathscr{H}_\infty$ LPV controller design): Let $\Psi(\alpha)$ parameterize a stabilizing full-order controller for the LPV system (1), for all $\alpha(\cdot) \in \mathscr{T}$, and let $\mathscr{A}_\Psi(\alpha)$, $\mathscr{B}_\Psi(\alpha)$, $\mathscr{C}_\Psi(\alpha)$ and $\mathscr{D}_\Psi(\alpha)$ denote the corresponding closed-loop matrices, as in (7). For a predefined controller order q $(0 \leq q < n_x)$, let U and V be given by

$$U = \begin{bmatrix} I_q & 0_{q\times(n_x-q)} & 0 \\ 0 & 0 & I_{n_u} \end{bmatrix}, \quad V = \begin{bmatrix} I_q & 0_{q\times(n_x-q)} & 0 \\ 0 & 0 & I_{n_y} \end{bmatrix}, \quad (9)$$

and let $A_{22}(\alpha) \in \mathbb{R}^{(n_x-q)\times(n_x-q)}$ be a given matrix that is exponentially stable for all $\alpha(\cdot) \in \mathscr{T}$. If there exist bounded matrices $P(\alpha) \in \mathbb{S}_+^{2n_x}$, $\bar{\Theta}(\alpha) \in \mathbb{R}^{(q+n_u)\times(n_x+n_y)}$ and

$$Y(\alpha) = \begin{bmatrix} Y_{11}(\alpha) & Y_{12}(\alpha) & Y_{13}(\alpha) \\ 0 & Y_{22}(\alpha) & 0 \\ Y_{31}(\alpha) & Y_{32}(\alpha) & Y_{33}(\alpha) \end{bmatrix}$$

with $Y_{11}(\alpha) \in \mathbb{R}^{q\times q}$, $Y_{22}(\alpha) \in \mathbb{R}^{(n_x-q)\times(n_x-q)}$, and $Y_{33}(\alpha) \in \mathbb{R}^{n_u\times n_u}$, for $k \geq 0$, and a scalar γ such that the LMI (12) holds for $\alpha(\cdot) \in \mathscr{T}$, where $Z(\alpha)$ is given by

$$Z(\alpha) := U'\bar{\Theta}(\alpha) + Y(\alpha)\left(\begin{bmatrix} 0_{q\times q} & 0 & 0 \\ 0 & A_{22}(\alpha) & 0 \\ 0 & 0 & 0_{n_u\times n_y} \end{bmatrix} - \Psi(\alpha)\right), \quad (10)$$

then the reduced-order LPV controller parameterized by

$$\Theta(\alpha) = \begin{bmatrix} Y_{11}(\alpha) & Y_{13}(\alpha) \\ Y_{31}(\alpha) & Y_{33}(\alpha) \end{bmatrix}^{-1} \bar{\Theta}(\alpha)V' \quad (11)$$

stabilizes the closed-loop system (5) with a guaranteed upper bound $\sqrt{\gamma}$ on its $\mathscr{H}_\infty$ performance for all parameter trajectories $\alpha(\cdot) \in \mathscr{T}$.

Proof: The proof is constructed by following the lines of the proof presented in [13] for LTI systems. ■

It is emphasized that all LMI variables in (12) can be chosen parameter-dependent. For instance, assuming a polynomial parameterization generally leads to a reduced-order controller with a rational parameter dependency, as implied by (11), while a polynomially parameter dependent controller results when $Y(\alpha)$ is taken constant. In addition, note that selecting $\bar{\Theta}(\alpha)$ constant corresponds to the synthesis of a robust LTI controller.

$$\begin{bmatrix} I & 0 & 0 \\ \mathscr{A}_\Psi(\alpha) & \mathscr{B}_\Psi(\alpha) & \tilde{B}_u(\alpha) \\ 0 & I & 0 \\ \mathscr{C}_\Psi(\alpha) & \mathscr{D}_\Psi(\alpha) & \tilde{D}_u(\alpha) \end{bmatrix}' \begin{bmatrix} -P(\alpha) & 0 & 0 & 0 \\ 0 & P(\alpha_+) & 0 & 0 \\ 0 & 0 & -\gamma I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ \mathscr{A}_\Psi(\alpha) & \mathscr{B}_\Psi(\alpha) & \tilde{B}_u(\alpha) \\ 0 & I & 0 \\ \mathscr{C}_\Psi(\alpha) & \mathscr{D}_\Psi(\alpha) & \tilde{D}_u(\alpha) \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} X_1(\alpha) \\ X_2(\alpha) \\ X_3(\alpha) \end{bmatrix} \begin{bmatrix} (\Theta_a(\alpha)-\Psi(\alpha))\tilde{C}_y(\alpha) & (\Theta_a(\alpha)-\Psi(\alpha))\tilde{D}_y(\alpha) & -I \end{bmatrix} \right\} \prec 0 \quad (8)$$

$$\begin{bmatrix} I & 0 & 0 \\ \mathscr{A}_\Psi(\alpha) & \mathscr{B}_\Psi(\alpha) & \tilde{B}_u(\alpha) \\ 0 & I & 0 \\ \mathscr{C}_\Psi(\alpha) & \mathscr{D}_\Psi(\alpha) & \tilde{D}_u(\alpha) \end{bmatrix}' \begin{bmatrix} -P(\alpha) & 0 & 0 & 0 \\ 0 & P(\alpha_+) & 0 & 0 \\ 0 & 0 & -\gamma I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ \mathscr{A}_\Psi(\alpha) & \mathscr{B}_\Psi(\alpha) & \tilde{B}_u(\alpha) \\ 0 & I & 0 \\ \mathscr{C}_\Psi(\alpha) & \mathscr{D}_\Psi(\alpha) & \tilde{D}_u(\alpha) \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} \begin{bmatrix} Z(\alpha)\tilde{C}_y(\alpha) & Z(\alpha)\tilde{D}_y(\alpha) & -Y(\alpha) \end{bmatrix} \right\} \prec 0 \quad (12)$$

When the LMI conditions are applied for multi-objective control design, all LMI variables except these responsible for controller reconstruction (i.e. $Y_{11}(\alpha), Y_{13}(\alpha), Y_{31}(\alpha), Y_{33}(\alpha)$, and $\bar{\Theta}(\alpha)$) can be chosen different for each performance specification.

Remark 2: The derivation of the synthesis conditions (12) relies on the specific selections $X_i(\alpha) = 0$ for $i = 1,2$ and $X_3(\alpha) = Y(\alpha)$ in the analysis LMI (8). These particular choices are necessary for reconstruction of a single LPV controller. As a consequence, H_Ψ should be exponentially stable and satisfy the $\mathscr{H}_\infty$ performance bound $\|H_\Psi\|_\infty^2 < \gamma$.

It is important to stress that the LMI condition (12) is semi-infinite, i.e. it should hold for all time instants $k \geq 0$, yielding an infinite number of constraints. A finite set of sufficient LMIs is derived by exploiting the structure of the $(\alpha, \Delta\alpha)$ domain, see Subsection IV-C.

IV. OVERHEAD CRANE APPLICATION

To demonstrate the practical applicability of the presented LMI approach, reduced-order $\mathscr{H}_2/\mathscr{H}_\infty$ controllers are designed for a lab-scale overhead crane, and numerically and experimentally compared with existing controller design approaches. The LMIs are implemented in MATLAB, using the software packages Yalmip [18] and SeDuMi [19].

A. Model Description

The system under consideration (depicted in Figure 1) consists of a velocity controlled cart on a rail, to which a load is attached through a cable with varying length. The horizontal cart and load position are denoted by x_{cart} [m] and x_{load} [m], respectively, while α [m] defines the cable length and θ [rad] is the swing angle. The system input is a voltage $v \in [-10, 10]$ [V], which scales to cart velocity through a high bandwidth velocity controller. The quantities x_{cart} and θ, as well as the varying cable length α, are measured online. To account for disturbance rejection in the control objective, an additional input d_θ is defined, modeling the effect of an initial swing angle disturbance. Specifically, selecting d_θ as

$$d_\theta(k) = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k > 0, \end{cases}$$

corresponds to an initial swing angle of 0.1rad. A multiple-input multiple-output 4th order LPV model with an affine dependency on α and a sampling period $T_s = 0.01s$ is identified using the SMILE technique [20], [6], and represented in state-space form as

$$G: \begin{cases} x(k+1) &= A(\alpha)x(k) + B(\alpha)u(k), \\ y(k) &= C(\alpha)x(k) + Du(k), \end{cases} \quad (13)$$

where the inputs and outputs are grouped in the vectors $u := \begin{bmatrix} v & d_\theta \end{bmatrix}'$ and $y := \begin{bmatrix} x_{\text{cart}} & \theta \end{bmatrix}'$. We select the set of admissible parameter trajectories as in (2), where Λ is the convex hull of

$$\left\{ \begin{bmatrix} \alpha_L \\ 0 \end{bmatrix}, \begin{bmatrix} \alpha_L \\ b \end{bmatrix}, \begin{bmatrix} \alpha_U - b \\ b \end{bmatrix}, \begin{bmatrix} \alpha_U \\ 0 \end{bmatrix}, \begin{bmatrix} \alpha_U \\ -b \end{bmatrix}, \begin{bmatrix} \alpha_L + b \\ -b \end{bmatrix} \right\}, \quad (14)$$

with $\alpha_L = 0.35$, $\alpha_U = 0.75$, and $b = 4 \cdot 10^{-3}$, corresponding to a cable length varying between 0.35m and 0.75m, and a maximum cable hoisting velocity of 0.4m/s. After specifying the control design objective in the next subsection, it is shown how the structure of this polytopic set is exploited to derive a finite set of sufficient LMIs for LPV controller synthesis in Subsection IV-C.

B. Control Design Objective

The goal is to design reduced-order LPV controllers of the form (3) for the identified LPV model (13), achieving a good tradeoff between reference tracking and rejection of swing angle disturbances under the influence of a varying cable length.

We define a reference signal r for the horizontal cart position, and a corresponding error signal $e := r - x_{\text{cart}}$. Ideally $e = 0$ and $\theta = 0$, hence the controller input is selected as $\begin{bmatrix} e & \theta \end{bmatrix}'$. To assure a high bandwidth and good reference

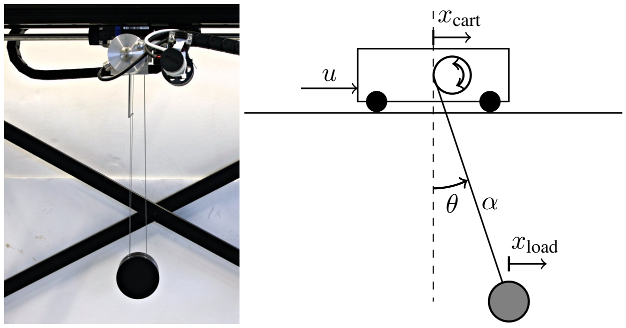

Fig. 1: The overhead crane setup (left) and its schematic representation (right).

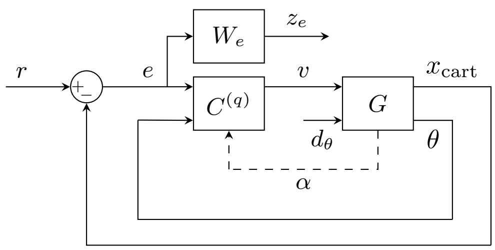

Fig. 2: Block diagram of the closed-loop system corresponding to the overhead crane model (13) interconnected with a dynamic output feedback LPV controller of order q. The exogenous input and regulated output are given by $[\, r \;\; d_\theta \,]'$, respectively, $[\, z_e \;\; \theta \,]'$.

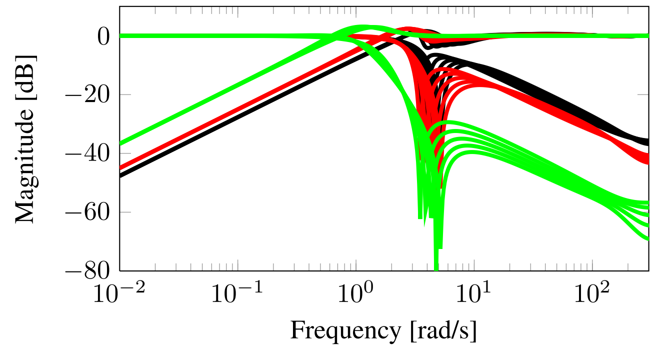

Fig. 3: Bode magnitude plots corresponding to the sensitivity $(r \to e)$ and complementary sensitivity $(r \to x_{\mathrm{cart}})$, evaluated at 5 equidistant cable lengths $\alpha \in [0.35, 0.75]$ for $b_{\mathscr{H}_2} = 1.2$ (black), $b_{\mathscr{H}_2} = 0.6$ (red) and $b_{\mathscr{H}_2} = 0.25$ (green).

tracking, we consider a weight function described by the continuous-time transfer function

$$W(s) := \frac{s/A_\infty + \omega_c}{s + A_0 \omega_c}, \tag{15}$$

where ω_c is the crossover frequency [rad/s], while $\lim_{s\to 0} W(s) = 1/A_0$ and $\lim_{s\to\infty} W(s) = 1/A_\infty$. Selecting $\omega_c = 0.2$, $A_0 = -60$dB and $A_\infty = 100$dB in (15), this transfer function is discretized using zero-order hold, resulting in the discrete-time LTI model $W_e : e \to z_e$. Figure 2 provides a schematic overview of the interconnected system. A $\mathscr{H}_\infty$ performance specification is selected for the channel $r \to z_e$ to assure a high bandwidth, while a $\mathscr{H}_2$ performance specification is imposed on the channel $d_\theta \to \theta$ to account for the rejection of swing angle disturbances. Our choice for a $\mathscr{H}_2$ performance stems from the fact that, in this case, minimization of the $\mathscr{H}_2$ norm relates to minimization of the *energy* in the system due to a swing angle disturbance.

C. Derivation of a Finite Set of Sufficient LMIs

Starting from a semi-infinite LMI constraint, the first step towards a finite set of sufficient LMIs is selecting a parameterization for all the LMI variables. We impose a polynomial parameter dependency on all LMI variables, resulting in a polynomially parameter-dependent LMI that should hold for all time instants $k \geq 0$. Applying the approach described in [21] (see also [4]), a finite set of sufficient LMIs is obtained by expressing all points in the $(\alpha, \Delta\alpha)$ domain in terms of the convex combination of the six vertices given in (14). In this new lifted domain with six parameters, the numerical complexity grows rapidly as the degree of the polynomial variables increase. To alleviate the computational burden, we propose a simplicial subdivision of the domain, such that any point is expressed in terms of solely three vertices. The benefits of such a subdivision for the involved control problem under consideration are illustrated below.

D. Full-order LPV Control

We employ the approach of [4] to design a strictly proper (i.e. $D_c(\alpha) = 0$) full-order $\mathscr{H}_2/\mathscr{H}_\infty$ LPV controller that achieves a good tradeoff between swing angle disturbance rejection and reference positioning. Applying a simplicial subdivision of the $(\alpha, \Delta\alpha)$ domain, the $\mathscr{H}_\infty$ performance is optimized subject to a fixed bound $b_{\mathscr{H}_2}$ on the $\mathscr{H}_2$ performance. An affine Lyapunov matrix is selected, resulting in a controller with a polynomial parameter dependency of degree 2. Subsequently, a full-order $\mathscr{H}_2/\mathscr{H}_\infty$ LPV controller is computed for different values of the bound $b_{\mathscr{H}_2}$. Figure 3 shows the Bode magnitude of the sensitivity $(r \to e)$ and complementary sensitivity $(r \to x_{\mathrm{cart}})$, evaluated at 5 equidistant cable lengths $\alpha \in [0.35, 0.75]$ for $b_{\mathscr{H}_2} = 1.2$ (black), $b_{\mathscr{H}_2} = 0.6$ (red) and $b_{\mathscr{H}_2} = 0.25$ (green). In a similar fashion, the Bode magnitude from d_θ to θ is shown in Figure 4. The gray lines indicate the open-loop transfer function ($b_{\mathscr{H}_2} = \infty$), featuring lightly damped resonance frequencies. More damped resonance frequencies correspond to better swing angle disturbance rejection, but this comes at the expense of a decrease in achievable bandwidth (see Figure 3). Based on the experimental responses, see Figure 7 in Subsection IV-F, we select the full-order controller corresponding to $b_{\mathscr{H}_2} = 0.6$ (i.e. the red lines in Figure 3 and Figure 4), and use this controller to design reduced-order multi-objective LPV controllers in the next subsection.

E. Reduced-order LPV Control

For the design of reduced-order LPV controllers, we select an affine parameter dependency for all LMI variables in the $\mathscr{H}_\infty$ synthesis LMIs (12) (and the associated $\mathscr{H}_2$ LMIs), and exploit the freedom to select different LMI variables for the

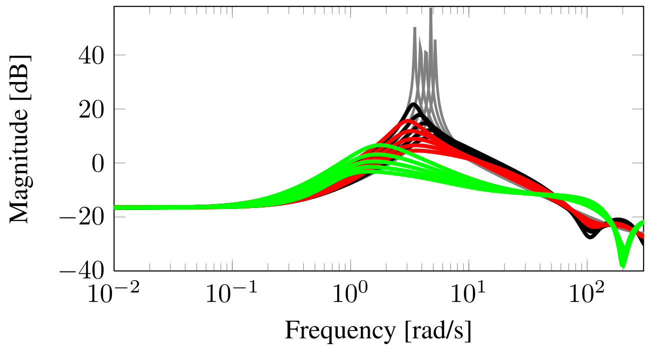

Fig. 4: Bode magnitude plots corresponding to the channel $d_\theta \to \theta$, evaluated at 5 equidistant cable lengths $\alpha \in [0.35, 0.75]$ for $b_{\mathscr{H}_2} = \infty$ (gray), $b_{\mathscr{H}_2} = 1.2$ (black), $b_{\mathscr{H}_2} = 0.6$ (red) and $b_{\mathscr{H}_2} = 0.25$ (green).

$\mathscr{H}_\infty$ and $\mathscr{H}_2$ performance (as explained after Theorem 2 in Section III). The full-order controller ($b_{\mathscr{H}_2} = 0.6$) computed in Subsection IV-D is used as parameter in the LMIs to compute controllers of order $q \in \{2,3,4\}$. As for the full-order case, the bandwidth is optimized while the bound $b_{\mathscr{H}_2} = 0.6$ on the $\mathscr{H}_2$ performance is maintained. To obtain a controller of order $q = 1$, a suboptimal full-order controller is computed by solving an LMI feasibility problem with the bounds $b_{\mathscr{H}_2} = 0.6$ and $b_{\mathscr{H}_\infty} = 0.5$ on the $\mathscr{H}_2$, respectively, $\mathscr{H}_\infty$ performance, and subsequently used in the reduced-order synthesis LMIs. No feasible result was obtained for $q = 0$. For each controller order, Table I provides an overview of the obtained $\mathscr{H}_\infty$ upper bound (see next paragraph for details), and the numerical complexity of the reduced-order synthesis LMIs. Note that the numerical burden is significantly reduced using subdivision, while the conservatism of the resulting $\mathscr{H}_\infty$ bounds is similar as compared to no subdivision.

Since numerical issues occur when solving the synthesis LMIs, reliable $\mathscr{H}_2$ and $\mathscr{H}_\infty$ performance bounds are computed a posteriori to validate the synthesized controllers, by solving a set of sufficient analysis LMIs (using subdivision of the $(\alpha, \Delta\alpha)$ domain). Since the reduced-order controllers have a rational parameter dependency, it is not straightforward to obtain a finite set of sufficient LMIs. Fortunately, the extended LMIs (8) (and their $\mathscr{H}_2$ counterparts) resolve this issue. Namely, applying the nonlinear change of variables

$$\tilde{X}_i(\alpha) := X_i(\alpha) \begin{bmatrix} Y_{11}(\alpha) & Y_{13}(\alpha) \\ Y_{31}(\alpha) & Y_{33}(\alpha) \end{bmatrix}^{-1}, \quad i = 1, \ldots, 3,$$

and choosing $\Psi(\alpha) = 0$, performance bounds are computed by inserting the polynomially parameter-dependent variables $\tilde{\Theta}(\alpha)$ and $Y(\alpha)$ instead of the reconstructed rationally parameter-dependent controller $\Theta(\alpha)$ in the LMIs (8). All the associated LMI variables are chosen to have a polynomial parameter dependency of degree 3, yielding the $\mathscr{H}_\infty$ upper bounds shown in Table I.

The upper bounds corresponding to the controllers of order $q \leq 3$ are considerably higher than the $\mathscr{H}_\infty$ upper bound corresponding to the full-order LPV controller. However, the Bode magnitude plots in Figure 5-6 indicate that the

TABLE I: For each order $q \in \{1,2,3,4\}$, the $\mathscr{H}_\infty$ bound and the number of LMI variables and LMI blocks (max. size = 28) corresponding to subdivision / no subdivision of the $(\alpha, \Delta\alpha)$ domain illustrate the merits of simplicial subdivision.

q	$\mathscr{H}_\infty$ bound subdiv. / no subdiv.	scalar LMI variables	LMI blocks subdiv. / no subdiv.
4	0.12 / 0.13	367	180 / 378
3	30.9 / 30.8	359	180 / 378
2	27.8 / 28.3	355	180 / 378
1	171 / 171	355	180 / 378

controller of order $q = 2$ has similar performance as the full-order controller for fixed cable lengths. Additionally, the experimental performances of these two controllers are comparable, as is shown in the next subsection (see Figure 7), motivating the practical use of the structurally simple controller.

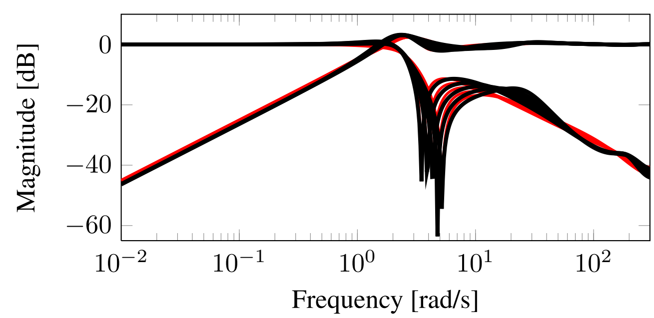

Fig. 5: Bode magnitude plots corresponding to the sensitivity $(r \to e)$ and complementary sensitivity $(r \to x_{\text{cart}})$, evaluated at 5 equidistant cable lengths $\alpha \in [0.35, 0.75]$ for the reduced-order (thick black) vs. the full-order controller (thin red) with $b_{\mathscr{H}_2} = 0.6$.

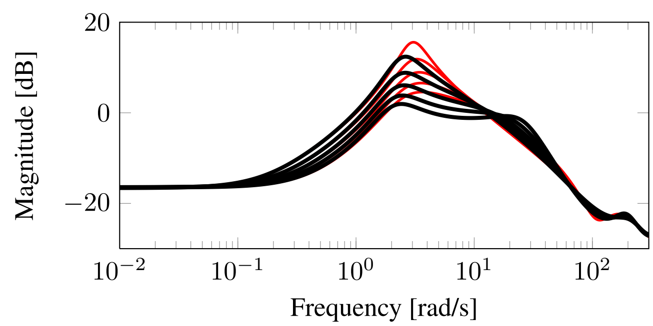

Fig. 6: Bode magnitude plots corresponding to the channel $d_\theta \to \theta$, evaluated at 5 equidistant cable lengths $\alpha \in [0.35, 0.75]$ for the reduced-order (thick black) vs. the full-order controller (thin red) with $b_{\mathscr{H}_2} = 0.6$.

F. Experimental Validation

The LPV controller of order $q = 2$ is implemented on the overhead crane test setup, and compared experimentally with

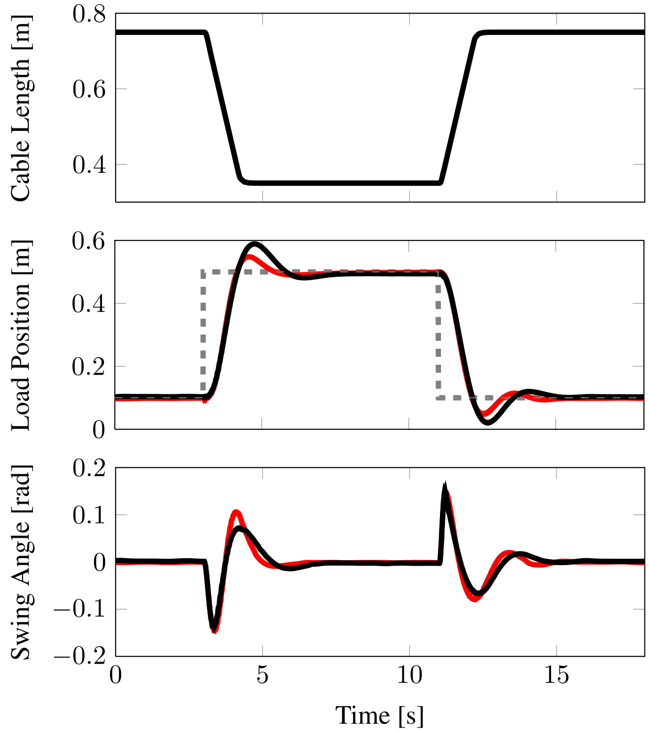

Fig. 7: The experimental results corresponding to the full-order (red) and reduced-order (black) controller yield similar closed-loop performances, motivating the practical use of the structurally simple controller.

the designed full-order LPV controller that satisfies the same performance bound $b_{\mathscr{H}_2} = 0.6$. A reference change of 0.4m is applied while the cable is hoisted at a rate of 0.4m/s. The latter corresponds to the maximum allowable rate of parameter variation for which the controller was designed. Figure 7 shows the measured cable length, load position and swing angle as a function of time. The closed-loop performances corresponding to the two controllers are similar, clearly showing the potential of the proposed reduced-order synthesis approach.

V. Conclusions

An LMI approach to design reduced-order multi-objective $\mathscr{H}_2/\mathscr{H}_\infty$ LPV controllers is presented, utilizing a full-order $\mathscr{H}_2/\mathscr{H}_\infty$ controller as parameter in sufficient LMIs for reduced-order control design. This paper extends our previous work [13] to LPV dynamics. By subdividing the domain where the parameter and its rate of variation assume values, the numerical burden is relieved while not loosing on conservatism, making our approach suitable for complex LMI problems. Its applicability to realistic engineering problems is illustrated by experimental validations.

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